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ON THE RELATION BETWEEN UPPER CENTRAL QUOTIENTS AND LOWER CENTRAL SERIES OF A GROUP

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ABSTRACT. Let H be a group with a normal subgroup N contained in the upper central subgroup Z_cH. In this article we study the influence of the quotient group $G = H/N$ on the lower central subgroup $\gamma_{c+1}H$. In particular, for any finite group G we give bounds on the order and exponent of $\gamma_{c+1}H$. For G equal to a dihedral group, or quaternion group, or extra-special group we list all possible groups that can arise as $\gamma_{c+1}H$. Our proofs involve: (i) the Baer invariants of G, (ii) the Schur multiplier $\mathcal{M}(L, G)$ of G relative to a normal subgroup L, and (iii) the nonabelian tensor product of groups. Some results on the nonabelian tensor product may be of independent interest.

1. Introduction

A group H gives rise to an upper central series $1 = Z_0H \leq Z_1H \leq \cdots$ and a lower central series $H = \gamma_1 H \geq \gamma_2 H \geq \cdots$. In this article we consider a normal subgroup $N \trianglelefteq H$ contained in Z_cH, and study the influence of the quotient $G = H/N$ on the lower central group $\gamma_{c+1}H$. An old result of R. Baer [1][18] states that $\gamma_{c+1}H$ is finite whenever the quotient G is finite. We develop Baer's techniques to obtain the following three results.

A. For any finite G we give an upper bound on the order of $\gamma_{c+1}H$. (A previous paper [9] gives a bound on $|\gamma_{c+1}H|$ when G is finite nilpotent; the present result incorporates a small improvement in this case. Several authors have given bounds when $c = 1$. In particular, there are papers by J.A. Green [17], J. Wiegold [29], [30], W. Gaschütz et al. [15], and M.R. Jones [19], [20], [21]. The case $c = 1$ is also studied in [11] where the results are slightly sharper than those obtained by specialising our general bound to $c = 1$.)

B. For any finite G we give an upper bound on the exponent of $\gamma_{c+1}H$. (For $c = 1$ this provides a generalisation of a result of A. Lubotzky and A. Mann [23] on the exponent of the Schur multiplier $\mathcal{M}(G)$ of a powerful p-group G; it also sharpens a bound of M.R. Jones [21] on the exponent of the Schur multiplier of a prime-power group. Furthermore, for $c \geq 1$ our bound yields a generalisation and sharpening of an estimate, given in [7], on the exponent of the c-nilpotent Baer invariant $M^{(c)}(G)$. This improvement for $c \geq 1$ has the following practical implication. An electronically down-loadable appendix to the paper [12] contains a MAGMA computer program for calculating a number of homotopy-theoretic constructions. In particular, it contains a function for computing $M^{(c)}(G)$ which requires, as input data, a finite presentation of a finite group G together with any positive integer q

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divisible by e^c where e denotes the exponent of $M^{(c)}(G)$. The improved estimate for e helps in choosing a suitable value for q.)

C. For G equal to a dihedral group, or quaternion group, or extra-special group we list all possible groups that can arise as $\gamma_{c+1}H$. (This extends the work of N.D. Gupta and M.R.R. Moghaddam [16] which handles the dihedral 2-groups. It also extends the work of D. MacHale and P.Ó'Murchú [26], and J. Burns et al. [8] which treats all groups G of order at most 30 for $c = 1$, and all groups G of order at most 16 for $c = 2$.)

A precise statement of results A-C is provided in Section 2. Their proofs are given in Sections 4-6 respectively. The proofs involve three techniques with which the reader may not be too familiar. The first is the use of a nonabelian tensor product of groups. The second is the use of a Schur multiplier $\mathcal{M}(N, G)$ of a group G relative to a normal subgroup N. The third is the use of Baer invariants of a group. Relevant details of these techniques are recalled, and developed, in Section 3. Some results in Section 3 (in particular Propositions 5, 8 and 9) may be of independent interest.

2. Statement of results

Let a group G be presented as the quotient of a free group F by a normal subgroup R. We state our results in terms of the Baer invariants

$$M^{(c)}(G) = \frac{R \cap \gamma_{c+1}F}{\gamma_{c+1}(R, F)}\ , \qquad c \geq 1,$$

and related invariants

$$\gamma^*_{c+1}(G) = \frac{\gamma_{c+1}F}{\gamma_{c+1}(R, F)}$$

of the group G, where $\gamma_1(R, F) = R$, $\gamma_{c+1}(R, F) = [\gamma_c(R, F), F]$ and $\gamma_{c+1}F = \gamma_{c+1}(F, F)$. It was shown by R. Baer [1] (see also [14] [25]) that these invariants are, up to group isomorphism, independent of the choice of free presentation of G. Note that there are canonical actions of G on $M^{(c)}(G)$ and $\gamma^*_{c+1}(G)$ given by conjugation in $F/\gamma_{c+1}(R, F)$.

If G is of the form $G \cong H/N$ with N a normal subgroup of H contained in Z_cH, then it is routine [6] to establish the existence of the canonical short exact sequence

$$A \rightarrowtail \gamma^*_{c+1}(G) \twoheadrightarrow \gamma_{c+1}H$$

where A is a submodule of the $\mathbf{Z}G$-module $M^{(c)}(G)$. We thus have inequalities (in which $\leq$ can be taken to mean 'divides')

$$|\gamma_{c+1}H| \leq |\gamma^*_{c+1}(G)| = |M^{(c)}(G)||\gamma_{c+1}G|\ , \tag{1}$$

$$\exp(\gamma_{c+1}H) \leq \exp(\gamma^*_{c+1}(G))\ , \tag{2}$$

involving the orders and exponents of groups. Since $M^{(c)}(G)$ is a subgroup of $\gamma^*_{c+1}(G)$, we also have an inequality

$$\exp(M^{(c)}(G)) \leq \exp(\gamma^*_{c+1}(G))\ . \tag{3}$$

Furthermore, a group K arises as $\gamma_{c+1}H$ for some H if and only if

$$K \cong \frac{\gamma^*_{c+1}(G)}{A} \tag{4}$$

with A a submodule of $M^{(c)}(G)$.

Observations (1), (2) and (4) allow us to state results A-C in terms of the invariant $\gamma_{c+1}^*(G)$. For the statement of result A we let $\chi_c(d)$ denote the number of elements in a basis of the free abelian group $\gamma_c F/\gamma_{c+1}F$ with F the free group on d generators. (There is a well-known formula for $\chi_c(d)$ due to E. Witt [27]. Let $\mu(m)$ be the Möbius function, defined for all positive integers by $\mu(1) = 1$, $\mu(p) = -1$ if p is a prime, $\mu(p^k) = 0$ for $k > 1$, and $\mu(ab) = \mu(a)\mu(b)$ if a and b are coprime integers. Witt's formula is

$$\chi_c(d) = (1/c) \sum_{m|c} \mu(m) d^{(c/m)}$$

where m runs through all divisors of c. Thus, for instance, $\chi_2(d) = (d^2 - d)/2$, $\chi_3(d) = (d^3 - d)/3$, $\chi_4(d) = (d^4 - d^2)/4$.)

For an arbitrary finite abelian p-group A we define the integer

$$\Lambda_c(A) = e_1 \chi_{c+1}(d_1) + \sum_{j=2}^{k} e_j \{\chi_{c+1}(d_1 + \cdots + d_j) - \chi_{c+1}(d_1 + \cdots + d_{j-1})\}$$

where the parameters d_j, e_j, k are determined by expressing A uniquely in the form

$$A \cong (C_{p^{e_1}})^{d_1} \times (C_{p^{e_2}})^{d_2} \times \cdots \times (C_{p^{e_k}})^{d_k}$$

with $e_1 > e_2 > \cdots > e_k \geq 1$.

For an arbitrary finite d-generator p-group P we define the integer

$$\Psi_c(P) = m_{c+1} d + m_c d^2 + \cdots + m_2 d^c$$

where the terms of the lower central series of P have orders $|\gamma_j(P)| = p^{m_j}$.

Note that an arbitrary finite group G has a smallest term L in its lower central series, namely the unique group $L = \gamma_r G$ that satisfies $\gamma_r G = \gamma_{r+1} G$. Suppose that P is a d-generator p-Sylow subgroup of G with Frattini subgroup $\Phi(P) = [P, P]P^p$, that $P/(P \cap [G, G])$ is a δ-generator group, that $(L \cap P)/(L \cap \Phi(P))$ has order p^t, that $L \cap P$ has order p^β, and that $[L \cap P, P]$ has order $p^{\beta'}$. We use these various parameters to define the integer

$$\Theta_c(G, L, P) = \beta + (\omega - \beta')(1 + \delta + \delta^2 + \cdots + \delta^{c-1}) \ ,$$

where

$$\omega = d\beta - (1/2)t(t + 1) \ .$$

Theorem A. *Let G be a finite group whose order has prime factors $p_1, p_2, \cdots, p_n$. Let L be the smallest term in the lower central series of G. The quotient G/L is nilpotent and thus a direct product*

$$G/L \cong S_1 \times S_2 \times \cdots \times S_n$$

with S_i a (possibly trivial) p_i-group. For each i let P_i be some p_i-Sylow subgroup of G. Then

$$|\gamma_{c+1}^*(G)| \leq \prod_{i=1}^{n} p_i^{\Lambda_c(S_i^{ab}) + \Psi_c(S_i) + \Theta_c(G, L, P_i)} \ .$$

The bound is attained, for instance, when G is abelian.

Note that if G is perfect, then $\Lambda_c(S_i^{ab}) = 0$, $\Psi_c(S_i) = 0$ and $\Theta_c(G, L, P_i) = d_i(2\beta_i - d_i - 1)/2 + \alpha_i$ where $p_i^{\beta_i}$ is the order of a d_i-generator p_i-Sylow subgroup P_i, and $p_i^{\alpha_i}$ is the order of the abelianisation P_i^{ab}. If, at the other extreme, G is nilpotent, then we have $\Theta_c(G, L, P_i) = 0$. If G is abelian, then $\Theta_c(G, L, P_i) = 0$ and $\Psi_c(S_i) = 0$.

The bound in Theorem A can be sharpened by involving the relative Schur multiplier $\mathcal{M}(L, G)$ whose definition is recalled in Section 3. More precisely, in the definition of $\Theta_c(G, L, P)$ we can redefine $\omega = \mu + \beta'$ where the pth primary component of the abelian group $\mathcal{M}(L, G)$ has order $|\mathcal{M}(L, G)_p| = p^\mu$. For example, if $|L|$ is coprime to $|G|/\exp(L)$, then the relative multiplier is trivial (see Proposition 7(ii)) and we can take $\Theta_c(G, L, P_i) = \beta_i$ for $c \geq 1$.

Before stating result B let us recall that A. Lubotzky and A. Mann [23] defined a p-group P to be *powerful* if: $p \geq 3$ and $[P, P] \subset P^p$; or $p = 2$ and $[P, P] \subset P^4$ (where P^i is the subgroup of P generated by all ith powers). In other words, P is powerful if $p \geq 3$ and P/P^p is abelian, or if $p = 2$ and P/P^4 is abelian. They proved a number of results about powerful groups P, one of which states that the exponent $\exp(M^{(1)}(P))$ of the Schur multiplier divides the exponent of P. We shall generalise this. Our generalisation implies, for instance, that $\exp(M^{(c)}(P))$ divides $\exp(P)$ for all $c \geq 1$ and all P in a certain class C_p of p-groups; the class C_p consists of those p-groups P satisfying $[[P^{p^{i-1}}, P], P] \subset P^{p^i}$ for $1 \leq i \leq e$ where $\exp(P) = p^e$. It is shown in [23] that if P is powerful, then $[P^{p^{i-1}}, P] \subset P^{p^i}$. Hence the class C_p contains all powerful p-groups.

Given a normal subgroup $N \trianglelefteq G$ of some group G, we say that the *pair* (N, G) has *nilpotency class* k if $\gamma_{k+1}(N, G) = 1$ and $\gamma_k(N, G) \neq 1$. For a real number r we let $[r]$ denote the smallest integer n such that $n \geq r$.

Let N be a normal subgroup of a finite p-group P and suppose that N has exponent p^e. We define the integer

$$\Omega(N, P) = [k_1/2] + [k_2/2] + \cdots + [k_e/2]$$

where k_j denotes the nilpotency class of the pair $(N^{p^{j-1}}/N^{p^j},\ P/N^{p^j})$ for $1 \leq j \leq e$. For N equal to the trivial group we set $\Omega(1, P) = 0$. Note that $\Omega(N, P) \leq [k/2]e$ where k is the nilpotency class of P.

Theorem B. *(i) Let G be a finite group whose order has prime factors $p_1, p_2, \cdots, p_n$. Let L be the smallest term in the lower central series of G. The quotient G/L is thus a direct product*

$$G/L \cong S_1 \times S_2 \times \cdots \times S_n$$

with S_i a (possibly trivial) p_i-group. For each i let P_i be a p_i-Sylow subgroup of G. Suppose that $L \cap P_i/[L \cap P_i, P_i]$ has exponent $p_i^{n_i}$. Suppose that the p_i-primary component of G^{ab} has exponent p^{e_i} with $e_i \geq 0$, and set $m_i = \min(\Omega(L \cap P_i, P_i), e_i)$. Then, for each $c \geq 1$,

$$\exp(\gamma^*_{c+1}(G)) \textit{ divides } \prod_{i=1}^{n} p_i^{\Omega(L\cap P_i, P_i) + \Omega(S_i, S_i) + n_i + (c-1)m_i}.$$

The bound is attained if G is abelian.

(ii) Suppose that a p-group P satisfies $[[P^{p^{i-1}}, P], P] \subset P^{p^i}$ for all $1 \leq i \leq e$ where p^e is the exponent of P. Then $\Omega(P, P) = e$.

Note that, by inequality (3), $\exp(M^{(c)}(G))$ divides $\exp(\gamma^*_{c+1}(G))$ for any group G. Thus, for an arbitrary finite p-group P of class k and exponent p^e, Theorem B(i) implies that $\exp(M^{(c)}(P))$ divides $p^{[k/2]e}$; this sharpens the bound $\exp(M^{(c)}(P)) \leq p^{(k-1)e}$ of Corollary 2.6 in [21] (for $c = 1$) and Theorem 6 in [7] (for $c \geq 1$). Theorem B(ii) implies that $\exp(\gamma^*_{c+1}(P))$ divides $\exp(P)$ if, for example, P is a p-group with P/P^p of nilpotency class 2 and P^p contained in the second centre $Z_2(P)$.

The bound in Theorem B(i) can be sharpened by redefining m_i to be $m_i = \min(\epsilon_i, e_i)$ where $p_i^{\epsilon_i}$ and p^{e_i} are the exponents of the p_i-primary components of $\mathcal{M}(L, G)$ and G^{ab} respectively (cf. Proposition 7 in Section 3). The bound is clearly independent of c if G is finite nilpotent, or if G is perfect. We do not know whether the bound can be made independent of c for arbitrary finite groups.

For the statement of result C we let $D_n = \langle a, b \mid a^2 = b^n = (ab)^2 = 1\rangle$ denote the dihedral group of order $2m$, and $Q_n = \langle a, b \mid a^2 = b^n = (ab)^2\rangle$ denote the quaternion group of order $4n$. Recall [3] that a p-group E is said to be *extra-special* if its commutator subgroup $[E, E]$, its Frattini subgroup $\Phi(E)$, and its centre $Z_1(E)$ coincide and have order p. The extra-special groups have order p^{2k+1} for $k \geq 1$, with precisely two extra-special p-groups for each k (see [3]). We let $E(p, k)$ denote an arbitrary extra-special p-group of order p^{2k+1}; we let $E(p, k)^+$ and $E(p, k)^-$ denote the extra-special p-groups of order p^{2k+1} and exponents p and p^2 respectively. For $c \geq 1$ we have the following.

Theorem C. *(i) For each $n \geq 2$ we have*

$$\gamma^*_{c+1}(D_n) \cong \begin{cases} C_n & \textit{odd } n, \\ C_n \times (C_2)^{\chi_{c+1}(2)-1} & \textit{even } n. \end{cases}$$

The generator $b \in D_n$ acts trivially on $\gamma^*_{c+1}(D_n)$; the generator $a \in D_n$ acts trivially on elements of order two, and inverts the elements of the cyclic summand C_n.

(ii) For each $n \geq 2$ we have

$$\gamma^*_{c+1}(Q_n) \cong \gamma^*_{c+1}(D_n)\,.$$

The generators $a, b \in Q_n$ act as in (i).

(iii) For each $k \geq 2$ we have

$$\gamma^*_{c+1}(E(p, k)) \cong (C_p)^{\chi_{c+1}(2k)}.$$

The group $E(p, k)$ acts trivially on $\gamma^*_{c+1}(E(p, k))$.

(iii)′ For $p \geq 3$ and some $1 \leq r \leq 2^c$ we have

$$\gamma^*_{c+1}(E(p, 1)^+) \cong (C_p)^{\chi_{c+1}(2)+r},$$

$$\gamma^*_{c+1}(E(p, 1)^-) \cong (C_p)^{\chi_{c+1}(2)}.$$

Note that the corresponding Baer invariants $M^{(c)}(G)$ are easily obtained applying the formula $M^{(c)}(G) = \ker(\gamma^*_{c+1}(G) \to \gamma_{c+1}(G))$ to the precise details of the isomorphisms given in the proof of Theorem C. This extends the computations on dihedral 2-groups given in [16]. (We remark that there is a slip in the statement of the main theorem in [16]; the statement is correct for $\gamma^*_{c+1}(D_{2^n})$ but incorrect for $M^{(c)}(D_{2^n})$.)

The precise value of r in Theorem C(iii)′ needs further investigation. The computer program listed in [12] yields the following results for the Burnside group $B(2, 3) = E(3, 1)^+$ of exponent 3 on two generators.

c	$\gamma^*_{c+1}(B(2,3))$	r
1	$(C_3)^3$	2
2	$(C_3)^5$	3
3	$(C_3)^9$	6
4	$(C_3)^{15}$	9
5	$(C_3)^{27}$	18

3. Preliminaries

The tensor product of nonabelian groups is a convenient setting for performing commutator calculations. Its functorial properties make it especially suited to the task of relating commutator calculations in a group to those in a homomorphic image of the group. We begin this section by recalling and developing relevant details on this tensor product. We then recall details on a Schur multiplier $\mathcal{M}(N,G)$ defined for pairs of groups. By a *pair* of groups (N,G) we simply mean a group G with normal subgroup N. The advantage of working with pairs is that any finite group G can be expressed as an extension

$$(L,G) \rightarrowtail (G,G) \twoheadrightarrow (G/L,G/L)$$

of a 'perfect' pair (L,G) by a 'nilpotent' pair $(G/L,G/L)$. Various simplifications apply when dealing with the Schur multiplier of perfect or nilpotent pairs. We end the section with some details on Baer invariants.

Suppose given two groups G and H which act on each other via group actions $G\times H \to H, (g,h)\mapsto {}^gh$ and $H\times G\to G, (h,g)\mapsto {}^hg$. Furthermore, suppose that each group acts on itself by conjugation, ${}^xy = xyx^{-1}$. (In keeping with this notation, our convention for commutators is $[x,y]=xyx^{-1}y^{-1}$.) The tensor product $G\otimes H$ is defined [5], [4] to be the group generated by symbols $g\otimes h$ subject to the relations

$$gg'\otimes h = ({}^gg'\otimes {}^gh)\,(g\otimes h)\,,$$

$$g\otimes hh' = (g\otimes h)\,({}^hg\otimes {}^hh')\,,$$

for $g,g'\in G, h,h'\in H$. The actions are said to be *compatible* if

$$^{({}^gh)}g' = {}^g({}^h({}^{g^{-1}}g'))\,,\ \ ^{({}^hg)}h' = {}^h({}^g({}^{h^{-1}}h'))$$

for all $g,g'\in G, h,h'\in H$.

Proposition 1 ([5])**.** *Suppose that G and H act compatibly on each other.*

(i) For all $g,g'\in G, h,h'\in H$ the following identities hold in $G\otimes H$:

$$g'\otimes ({}^gh)\,h^{-1} = {}^{g'}(g\otimes h)\,(g\otimes h)^{-1}\,,$$

$$g\,({}^hg^{-1})\otimes h' = (g\otimes h)\,{}^{h'}(g\otimes h)^{-1}\,,$$

$$g\,({}^hg^{-1})\otimes ({}^{g'}h')\,h'^{-1} = [g\otimes h, g'\otimes h']\,.$$

(ii) There is a homomorphism $\partial_G\colon G\otimes H\to G, g\otimes h\mapsto g\,{}^hg^{-1}$.

(iii) There is a 'diagonal' action of G on $G\otimes H$ given by ${}^{g'}(g\otimes h) = ({}^{g'}g\otimes {}^{g'}h)$.

(iv) There is an isomorphism $G\otimes H\xrightarrow{\cong} H\otimes G,\ g\otimes h\mapsto h\otimes g$.

(v) If ${}^g h = h$, ${}^h g = g$ *for all* $g \in G, h \in H$, *then* $G \otimes H \cong G^{ab} \otimes H^{ab}$, *where the right-hand side of the isomorphism denotes the usual tensor product of abelian groups.*

For each pair of groups (N, G) we can form the tensor product $N \otimes G$ in which all actions are taken to be conjugation in G. Since conjugation yields compatible actions, there is a diagonal action of G on $N \otimes G$. The tensor product $N \otimes G$ acts on G by conjugation in G via the homomorphism $\partial_N \colon N \otimes G \to N$. We can thus construct the triple tensor product $(N \otimes G) \otimes G$. One readily checks that the construction preserves 'compatibility of actions', and that it can therefore be iterated to form the $(c+1)$-fold tensor product

$$\bigotimes^{c+1}(N, G) = (\cdots((N \otimes G) \otimes G) \otimes \cdots \otimes G), \qquad c \geq 1,$$

involving c copies of G and one copy of N.

Proposition 2 ([13])**.** *Let* G *be a* d*-generator* p*-group with normal subgroup* N*. Suppose that* $|\gamma_i(N, G)| = p^{m_i}$ *for* $i \geq 1$, $m_i \geq 0$*. Then, for any* $c \geq 1$*, we have*

$$|\bigotimes^{c+1}(N, G)| \leq p^{m_c d + m_{c-1} d^2 + \cdots + m_1 d^c}\,.$$

Lemma 3. *Let* $G_3 \hookrightarrow G_2 \twoheadrightarrow G_1$*,* $H_3 \hookrightarrow H_2 \twoheadrightarrow H_1$ *be two short exact sequences of groups. Suppose that* G_i *and* H_i *act compatibly on one another for* $1 \leq i \leq 3$*, and that the homomorphisms preserve actions. Then there is an exact sequence of homomorphisms*

$$(G_3 \otimes H_2) \overline{\times} (G_2 \otimes H_3) \longrightarrow G_2 \otimes H_2 \longrightarrow G_1 \otimes H_1 \longrightarrow 1$$

in which $\overline{\times}$ *denotes a semi-direct product whose details need not be specified.*

Proof. The lemma is a routine adaption of Proposition 9 in [4]. □

Lemma 4. *Let* N *be a normal subgroup of* G *for which the commutator* $[n, [n, g]]$ *is trivial for all* $g \in G, n \in N$*. In the tensor product* $N \otimes G$*, with* G *and* H *acting by conjugation, the following identity holds for all* $g \in G, n \in N$ *and all integers* $t \geq 2$*:*

$$n^t \otimes g = (n \otimes g)^t \, (n \otimes [n, g]^{t(t-1)/2})\,.$$

Proof. The case $N = G$ is proved in [2]. The proof of the more general case is analogous; it can also be derived directly using Proposition 1(i). □

Recall that a pair (N, G) is said to be *nilpotent* of *class* k if $\gamma_{k+1}(N, G) = 1$ and $\gamma_k(N, G) \neq 1$. Also recall that $[k/2]$ denotes the smallest integer n such that $n \geq k/2$.

Proposition 5. *Let* G *be a group with normal subgroup* N*. Suppose that* N *has prime-power exponent* p^e *and that the pair* (N, G) *has nilpotency class* $\leq k$*. Then, for any* $c \geq 1$*, we have*

$$\exp(\bigotimes^{c+1}(N, G)) \text{ divides } p^{[k/2]e}.$$

Proof. For $t = p^e$ the binomial coefficient $\binom{t}{2}$ is divisible by t when $p \geq 3$, and divisible by $t/2$ when $p = 2$. Thus Lemma 4 proves the proposition for $k = 2, c = 1$ (since for $p = 2$, $t = p^e$ and $\gamma_3(N,G) = 1$ the identity

$$n \otimes [n,g]^{t/2} = n \otimes [n^{t/2}, g]$$

holds for all $g \in G, n \in N$; but $[n^{t/2}, g] = 1$ because $n^{t/2}$ has order at most 2.)

Let us now consider $k = 2$ and some $c \geq 2$. Then

$$\bigotimes^{c+1}(N,G) = \bigotimes^{c}(N \otimes G, G)$$

and $N \otimes G$ acts trivially on G. The triviality of this action implies the identity

$$(\cdots((n \otimes g)^t \otimes g_1) \otimes \cdots \otimes g_c) = (\cdots((n \otimes g) \otimes g_1) \otimes \cdots \otimes g_c)^t$$

in $\bigotimes^{c+1}(N,G)$. Hence $\exp(\bigotimes^{c+1}(N,G))$ divides $\exp(N \otimes G)$ and the proposition is proved for $k = 2, c \geq 1$.

Suppose now that the proposition has been proved for some c and all $k < k_0$. Suppose $\gamma_{k_0+1}(N,G) = 1$. Lemma 3 implies an exact sequence

$$(\gamma_{k_0-1}(N,G) \otimes G)\overline{\times}(N \otimes \gamma_{k_0-1}(N,G)) \to N \otimes G \to \frac{N}{\gamma_{k_0-1}(N,G)} \otimes \frac{G}{\gamma_{k_0-1}(N,G)}\,.$$

Working in $N \otimes G$, the image of $\gamma_{k_0-1}(N,G) \otimes G$ contains the image of $N \otimes \gamma_{k_0-1}(N,G)$ by virtue of the identity

$$m \otimes [n,g] = ([n,g] \otimes m)^{-1}$$

which follows from Proposition 1(i) for all $g \in G, m, n \in N$. We thus have an exact sequence

$$\gamma_{k_0-1}(N,G) \otimes G \to N \otimes G \to \frac{N}{\gamma_{k_0-1}(N,G)} \otimes \frac{G}{\gamma_{k_0-1}(N,G)}\,.$$

By applying Lemma 3 to this sequence, and invoking a similar identity, we obtain the exact sequence

$$\begin{aligned}(\gamma_{k_0-1}(N,G) \otimes G) \otimes G &\to (N \otimes G) \otimes G \\ &\to \left(\frac{N}{\gamma_{k_0-1}(N,G)} \otimes \frac{G}{\gamma_{k_0-1}(N,G)}\right) \otimes \frac{G}{\gamma_{k_0-1}(N,G)}\,.\end{aligned}$$

Repetition of the process yields an exact sequence

$$\bigotimes^{c+1}(\gamma_{k_0-1}(N,G), G) \to \bigotimes^{c+1}(N,G) \to \bigotimes^{c+1}\left(\frac{N}{\gamma_{k_0-1}(N,G)}, \frac{G}{\gamma_{k_0-1}(N,G)}\right)$$

from which we deduce that $\exp(\bigotimes^{c+1}(N,G)) \leq p^{[(k_0-2)/2]}p^e = p^{[k_0/2]}$. By induction, the proposition is proved for all $c, k \geq 1$. □

Following J.-L.Loday [22] we say that a pair of groups (N,G) is *perfect* if $N = [N,G]$.

Proposition 6. *Let (N,G) be any perfect pair of groups and set*

$$M = \ker(\partial_N \colon N \otimes G \to N).$$

Then M is abelian and, for each $c \geq 1$, there is an exact sequence

$$\bigotimes^{c+1}(M, G^{ab}) \to \bigotimes^{c+2}(N,G) \to \bigotimes^{c+1}(N,G) \to 1$$

where $\bigotimes^{c+1}(M, G^{ab})$ *is the usual iterated tensor product of abelian groups.*

Proof. Let $H_3 \hookrightarrow H_2 \twoheadrightarrow H_1$ be a short exact sequence of groups, and let G be a group such that G and H_i act compatibly on each other for $1 \le i \le 3$ with the homomorphisms preserving the actions. Then H_3 acts trivially on G via H_2. Suppose that the action of G on H_2 restricts to a trivial action of G on H_3. Then Proposition 1(v) and Lemma 3 imply an exact sequence

$$H_3^{ab} \otimes G^{ab} \to H_2 \otimes G \to H_1 \otimes G \to 1. \tag{5}$$

A perfect pair of groups (N, G) gives rise to a short exact sequence $\ker(\partial_N) \hookrightarrow N \otimes G \overset{\partial}{\twoheadrightarrow} N$. The identity

$$^{h}(n \otimes g) = ([n, g] \otimes h)^{-1} (n \otimes g)$$

which holds in $N \otimes G$ (see Proposition 1) for all $g, h \in G, n \in N$ implies that G acts trivially on $\ker(\partial_N)$. So (5) implies an exact sequence

$$M \otimes G^{ab} \to (N \otimes G) \otimes G \xrightarrow{\partial \otimes 1} N \otimes G \to 1\,.$$

Note that the diagonal action of G on $M \otimes G^{ab}$ is trivial, and hence G acts trivially on $\ker(\partial \otimes 1)$. Thus a second application of (5) yields the exact sequence $\ker(\partial \otimes 1) \otimes G^{ab} \to \bigotimes^4(N, G) \to \bigotimes^3(N, G) \to 1$. From this we derive the exact sequence $\bigotimes^3(M, G^{ab}) \to \bigotimes^4(N, G) \to \bigotimes^3(N, G) \to 1$. The proposition follows from a repetition of this argument. □

Given a pair of groups (N, G) we denote by $\Delta(N, G)$ the subgroup of $N \otimes G$ generated by the elements $n \otimes n$ for $n \in N$. This is a normal subgroup and following [5] we define the *exterior product*

$$N \wedge G = N \otimes G / \Delta(N, G)\,.$$

The homomorphism $\partial_N \colon N \otimes G \to N$ clearly induces a homomorphism $\partial_N \colon N \wedge G \to N$. The identity

$$[g, n] \otimes [g', n'] = [(g \otimes n), (g' \otimes n')]\,,$$

of Proposition 1(i) implies an isomorphism $N \wedge G \cong N \otimes G$ in the case of perfect pairs.

Definition ([10])**.** The *Schur multiplier* of a pair of groups (N, G) is the group $\mathcal{M}(N, G)$ defined by

$$\mathcal{M}(N, G) = \ker(\partial_N \colon N \wedge G \to N)\,.$$

If the pair is perfect, then, equivalently,

$$\mathcal{M}(N, G) = \ker(\partial_N \colon N \otimes G \to N)\,.$$

Proposition 7 ([10])**.** *Let G be a finite group with normal subgroup $N \trianglelefteq G$.*

(i) Then $\mathcal{M}(N, G)$ is a finite abelian group with exponent e dividing the order of G.

(ii) Let e' denote the exponent of N. Then, in fact, ee' divides the order of G and e divides the order of N.

(iii) Let K be any subgroup of G such that each $g \in G$ can be expressed (not necessarily uniquely) as a product $g = nk$ with $n \in N, k \in K$. Then e^2 divides $|N| \times |K|$.

(iv) Suppose that P is a p-Sylow subgroup of G. Let $\mathcal{M}(P\cap N,P)_p$ denote the p-component of the multiplier, and $\iota\colon \mathcal{M}(P\cap N,P)\to\mathcal{M}(N,G)$ the homomorphism induced by inclusion. Then

$$\mathcal{M}(P\cap N,P)\cong\mathcal{M}(N,G)_p\oplus\ker(\iota)\,.$$

Proposition 8. *Let G be a δ-generator p-group with normal subgroup $N\trianglelefteq G$ of order $|N|=p^\beta$. Suppose that $|N/(N\cap\Phi(G))|=p^t$ where $\Phi(G)=[G,G]G^p$. Then*

$$|N\wedge G|\le p^{\delta\beta-\frac{t(t+1)}{2}}\,.$$

Proof. Proposition 2 implies that $|N\otimes G|\le p^{\delta\beta}$. There is a commutative diagram of group homomorphisms

$$\begin{tikzcd}
& \Gamma(\frac{N}{[N,G]}) \arrow[r, "\Delta'"] \arrow[dd] & N\otimes G \arrow[r] \arrow[d] & N\wedge G \arrow[r] & 1 \\
& & G\otimes G \arrow[d, two heads] & & \\
1 \arrow[r] & \Gamma(\frac{G}{\Phi(G)}) \arrow[r, "\Delta"] & \frac{G}{\Phi(G)}\otimes\frac{G}{\Phi(G)} \arrow[r] & \frac{G}{\Phi(G)}\wedge\frac{G}{\Phi(G)} \arrow[r] & 1
\end{tikzcd}$$

in which the rows (but not the columns) are exact [5]. The abelian group $\Gamma(A)$, defined for any additive abelian group A, is J.H.C. Whitehead's universal quadratic construction; it is generated (as an abelian group) by symbols $\gamma(a)$ for $a\in A$ subject to the relations

$$\gamma(-a)=\gamma(a),$$

$$\gamma(a+b+c)+\gamma(a)+\gamma(b)+\gamma(c)=\gamma(a+b)+\gamma(a+c)+\gamma(b+c)$$

for $a,b,c\in A$. The homomorphism Δ is defined on generators by $\Delta(\gamma(x))=x\otimes x$ for $x\in G/\Phi(G)$. The image of $\Gamma(G/[N,G])$ in $G/\Phi(G)\otimes G/\Phi(G)$ is an elementary abelian group of rank $t(t+1)/2$. Hence the exactness of the top row implies

$$|N\wedge G|=\frac{|N\otimes G|}{|\Delta'(\Gamma(N/[N,G]))|}\le\frac{p^{\delta\beta}}{p^{t(t+1)/2}}\,.$$

This proves the proposition. ◻

Proposition 9. *Let N be a normal subgroup of a group G. If N has exponent p^e, then*

$$\exp(N\wedge G)\ \textit{divides}\ p^{[k_1/2]+[k_2/2]+\cdots+[k_e/2]}$$

where k_i denotes the nilpotency class of the pair $(N^{p^{i-1}}/N^{p^i},\,G/N^{p^i})$ for $1\le i\le e$.

Proof. Let K,M be normal subgroups of G with $K\le M$. Using the identity $m\otimes k=(k\otimes m)^{-1}$ which holds in $M\wedge G$ for all $k\in K, m\in M$, one readily develops the short exact sequence

$$K\wedge G\to M\wedge G\to M/K\wedge G/K\to 1 \tag{6}$$

from Lemma 3. Now (6) yields the exact sequences

$$N^{p^i}\wedge G\to N^{p^{i-1}}\wedge G\to N^{p^{i-1}}/N^{p^i}\wedge G/N^{p^i}\to 1$$

for $i \geq 1$. Hence

$$\exp(N \wedge G) \leq \prod_{i=1}^{e} \exp(N^{p^{i-1}}/N^{p^i} \wedge G/N^{p^i}) \,.$$

Proposition 5 implies $\exp(N^{p^{i-1}}/N^{p^i} \wedge G/N^{p^i}) \leq p^{[k_i/2]}$. □

Tensor products are related to Baer invariants by the following slight generalisation of a result of A.-S.T. Lue [24] (cf. [6]).

Proposition 10 ([24])**.** *For any group G with normal subgroup $N \trianglelefteq G$, and for $c \geq 1$, there is an exact sequence*

$$\bigotimes^{c+1}(N, G) \to \gamma^*_{c+1}(G) \to \gamma^*_{c+1}(G/N) \to 1 \,.$$

It is convenient to set

$$\overline{\gamma}^*_{c+1}(N, G) = \ker(\gamma^*_{c+1}(G) \to \gamma^*_{c+1}(G/N)) \,.$$

Note that $\gamma^*_{c+1}(G) = \overline{\gamma}^*_{c+1}(G, G)$. (The bar is intended to suggest that $\overline{\gamma}^*_{c+1}(N, G)$ is a quotient of some functor $\gamma^*_{c+1}(N, G)$. For example, we can take $\gamma^*_2(N, G) = N \wedge G$ [6]. Proposition 9 and Proposition 11 could be subsumed under a single result concerning $\gamma^*_{c+1}(N, G)$.)

Proposition 11. *Suppose that N is a normal subgroup of a group G. If N has exponent p^e, then*

$$\exp(\overline{\gamma}^*_{c+1}(N, G)) \textit{ divides } p^{[k_1/2]+[k_2/2]+\cdots+[k_e/2]}$$

where k_i denotes the nilpotency class of the pair $(N^{p^{i-1}}/N^{p^i},\, G/N^{p^i})$ for $1 \leq i \leq e$.

Proof. The proof is analogous to that of Proposition 9, but with (6) replaced by the exact sequence

$$\overline{\gamma}^*_{c+1}(K, G) \to \overline{\gamma}^*_{c+1}(M, G) \to \overline{\gamma}^*_{c+1}(M/K, G/K) \to 1 \,.$$

We leave the verification of the exactness of this (canonical) sequence to the reader. □

The *upper epicentral series* of an arbitrary group G was introduced in [6]. This is a family of characteristic subgroups $1 = Z^*_0(G) \leq Z^*_1(G) \leq Z^*_2(G) \leq \cdots$ with various useful properties such as those listed in the next proposition. Part (i) of the following proposition can be taken as the definition of $Z^*_c(G)$.

Proposition 12 ([6])**.** *Let $c \geq 1$.*

(i) $Z^*_c(G)$ is the smallest normal subgroup of G, contained in $Z_c(G)$, such that the quotient $G/Z^*_c(G)$ is isomorphic to H/Z_cH for some group H.

(ii) $Z^*_{c+1}(G)$ contains $Z^*_c(G)$.

(iii) $Z^*_c(G) = 1$ if and only if there exists an isomorphism $G \cong H/Z_cH$ for some group H.

(iv) Let N be a normal subgroup of G. Then $N \leq Z^*_c(G)$ if and only if the quotient homomorphism $G \twoheadrightarrow G/N$ induces an isomorphism $\gamma^*_{c+1}(G) \xrightarrow{\cong} \gamma^*_{c+1}(G/N)$.

Let A be a d-generator abelian group with generators $a_1, \cdots, a_d$. Let A_i denote the cyclic subgroup of A generated by a_i. Let $\mathcal{L}(d)$ denote the set of basic commutators on the d symbols a_i. To each basic commutator $\lambda = [a_{i_1}, \cdots, a_{i_k}]$ of weight k we associate the k-fold tensor product of abelian groups $T(\lambda) = A_{i_1} \otimes \cdots \otimes A_{i_k}$.

Thus T is a cyclic group of order equal to the highest common factor of the orders of the A_{i_j}. It is explained in [9] that the invariant $\gamma^*_{c+1}(A)$ is isomorphic to a direct sum of cyclic groups

$$\gamma^*_{c+1}(A) \cong \bigoplus_{\lambda\in\mathcal{L}(d)} T(\lambda)\,.$$

The following proposition is an immediate corollary to this isomorphism. An alternative derivation of the proposition can be found in [28].

Proposition 13. *Let A be a direct product of cyclic groups*

$$A = (C_{n_1})^{d_1} \times (C_{n_2})^{d_2} \times \cdots \times (C_{n_k})^{d_k}$$

with each n_i divisible by n_{i+1}. Then

$$\gamma^*_{c+1}(A) \cong (C_{n_1})^{\chi_{c+1}(d_1)} \times \prod_{j=2}^{k} (C_{n_j})^{\{\chi_{c+1}(d_1+\cdots+d_j)-\chi_{c+1}(d_1+\cdots+d_{j-1})\}}\,.$$

Proposition 14 ([9])**.** *Let $G = S_1 \times S_2 \times \cdots \times S_k$ be a direct product of groups whose abelianisations S_i^{ab} have finite, and mutually coprime, orders. Then, for each $c \geq 1$, there is an isomorphism*

$$\gamma^*_{c+1}(G) \cong \gamma^*_{c+1}(S_1) \times \cdots \times \gamma^*_{c+1}(S_k)\,.$$

Proposition 15. *Let N be a nontrivial normal subgroup of a p-group G. Let K denote the kernel of the canonical surjection $\gamma^*_{c+1}(G) \twoheadrightarrow \gamma^*_{c+1}(G/N)$.*

(i) K is nontrivial if and only if there exists some group H for which $H/Z_cH \cong G$.

(ii) If $p \geq 3$ and $N \subset G^p \cap Z_2G$, or if $p = 2$ and $N \subset G^{p^2} \cap Z_2G$, then K is contained in the Frattini subgroup of $\gamma^*_{c+1}(G)$.

(iii) If N is a proper subgroup of a cyclic normal subgroup in G, and if $N \subset Z_2G$, then K is contained in the Frattini subgroup of $\gamma^*_{c+1}(G)$.

Proof. Proposition 12 implies (i).

Proposition 10 implies that K is generated by the image of tensors of the form $(\cdots((n \otimes g_1) \otimes g_2) \otimes \cdots \otimes g_c)$. The hypothesis of (ii) with Lemma 4 implies that the canonical image in $\bigotimes^{c+1}(G,G)$ of each such tensor lies in the subgroup $\bigotimes^{c+1}(G,G)^p$ generated by pth powers of tensors. The hypothesis of (iii) implies that the image lies in the subgroup generated by pth powers of tensors together with tensors of the form $(\cdots((g \otimes g^t) \otimes g_2) \otimes \cdots \otimes g_c)$. In both cases K lies in the Frattini subgroup of the p-group $\gamma^*_{c+1}(G)$. ∎

4. Proof of Theorem A

Let G, S_i, P_i, L be as in the statement of Theorem A. For each prime p_i let $\bigotimes^{c+1}(L,G)_{p_i}$ denote some p_i-Sylow subgroup of $\bigotimes^{c+1}(L,G)$, and set

$$\Lambda^i_c = \log_{p_i} |\gamma^*_{c+1}(S_i^{ab})|\,,$$

$$\Psi^i_c = \log_{p_i} |\bigotimes^{c+1}([S_i,S_i],S_i)|\,,$$

$$\Theta^i_c = \log_{p_i} |\bigotimes^{c+1}(L,G)_{p_i}|\,.$$

Propositions 10 and 14 imply exact sequences

$$\bigotimes^{c+1}(L,G) \to \gamma^*_{c+1}(G) \to \prod_{i=1}^{n} \gamma^*_{c+1}(S_i) \to 1\,,$$

$$\bigotimes^{c+1}([S_i,S_i],S_i) \to \gamma^*_{c+1}(S_i) \to \gamma^*_{c+1}(S_i^{ab}) \to 1\,.$$

Hence

$$|\gamma^*_{c+1}(G)| \le \prod_{i=1}^{n} p_i^{\Lambda_c^i+\Psi_c^i+\Theta_c^i}\,.$$

To complete the proof we must find appropriate upper bounds $\Lambda_c(S_i^{ab})$, $\Psi_c(S_i)$, $\Theta_c(G,L,P_i)$ for Λ_c^i, Ψ_c^i, Θ_c^i.

Proposition 13 furnishes the appropriate formula for $\Lambda_c(S_i^{ab})$. Proposition 2 provides the appropriate formula for $\Psi_c(S_i)$. Suppose that P_i is a d_i-generator group, that $P_i/(P_i \cap [G,G])$ is a δ_i-generator group, that $(L\cap P_i)/(L\cap\Phi(P_i))$ has order p^{t_i}, that $L\cap P_i$ has order $p_i^{\beta_i}$, and that $[L\cap P_i,P_i]$ has order $p^{\beta_i'}$. Set $M=\mathcal{M}(L,G)=\ker(L\otimes G \twoheadrightarrow L)$ and let M_{p_i} denote the p_i-primary component of M. Since the pair (L,G) is perfect, Proposition 6 implies

$$\begin{aligned}|\bigotimes^{c+1}(L,G)| \le |\bigotimes^{c}(M,G^{ab})| \times |\bigotimes^{c-1}(M,G^{ab})| \times \cdots \\ \times |\bigotimes^{2}(M,G^{ab})| \times |M| \times |L|\,,\end{aligned}$$

and thus

$$\begin{aligned}|\bigotimes^{c+1}(L,G)_{p_i}| \le |\bigotimes^{c}(M_{p_i},G^{ab})| \times |\bigotimes^{c-1}(M_{p_i},G^{ab})| \times \cdots \\ \times |\bigotimes^{2}(M_{p_i},G^{ab})| \times |M_{p_i}| \times |L\cap P_i|\,.\end{aligned}$$

Proposition 7(iv) implies that $M_{p_i} \subset \mathcal{M}(L\cap P_i,P_i)$. Since $|L\cap P_i| = p_i^{\beta_i}$, Proposition 8 implies that

$$|\mathcal{M}(L\cap P_i,P_i)| \times |[L\cap P_i,P_i]| \le p_i^{d_i\beta_i - \frac{t_i(t_i+1)}{2}}\,.$$

Setting $\omega_i = d_i\beta_i - (1/2)t_i(t_i+1)$, we have

$$|M_{p_i}| \le |\mathcal{M}(L\cap P_i,P_i)| \le p_i^{\omega_i-\beta_i'}$$

and hence

$$|\bigotimes^{c+1}(L,G)_{p_i}| \le p_i^{(\omega_i-\beta_i')\delta_i^c} p_i^{(\omega_i-\beta_i')\delta_i^{(c-1)}} \cdots p_i^{(\omega_i-\beta_i')\delta_i} p_i^{(\omega_i-\beta_i')} p_i^{\beta_i}\,.$$

This yields the appropriate formula for $\Theta_c(G,L,P_i)$.

5. Proof of Theorem B

Let G, S_i, P_i, L be as in the statement of Theorem B. Let $\overline{\gamma}^*_{c+1}(L,G)_{p_i}$ denote a p_i-Sylow subgroup of the group $\overline{\gamma}^*_{c+1}(L,G)$. The short exact sequence $\overline{\gamma}^*_{c+1}(L,G) \hookrightarrow \gamma^*_{c+1}(G) \twoheadrightarrow \gamma^*_{c+1}(G/L)$ with Proposition 14 implies

$$\exp(\gamma^*_{c+1}(G)) \leq \prod_{i=1}^{n} \exp(\overline{\gamma}^*_{c+1}(L,G)_{p_i})\ \exp(\gamma^*_{c+1}(S_i))\,.$$

We apply Propositions 6 and 10 and the surjection $\bigotimes^{c+1}(L,G) \twoheadrightarrow \overline{\gamma}^*_{c+1}(L,G)$ to obtain

$$\exp(\gamma^*_{c+1}(G)) \leq \prod_{i=1}^{n} \exp((\mathcal{M}(L,G)\otimes G^{ab})_{p_i})^{c-1}\ \exp((L\otimes G)_{p_i})\ \exp(\gamma^*_{c+1}(S_i))\,.$$

Proposition 7(iv) yields

$$\exp(\gamma^*_{c+1}(G)) \leq \prod_{i=1}^{n} \exp(\mathcal{M}(L\cap P_i,P_i)\otimes G^{ab})^{c-1}\ \exp((L\otimes G)_{p_i})\ \exp(\gamma^*_{c+1}(S_i))\,.$$

The exact sequence

$$(L\cap P_i)\wedge P_i \to (L\otimes G)_{p_i} \to L\cap P_i/[L\cap P_i,P_i] \to 1$$

is readily derived, and yields

$$\exp(\gamma^*_{c+1}(G)) \leq \prod_{i=1}^{n} \exp(\mathcal{M}(L\cap P_i,P_i)\otimes G^{ab})^{c-1}\ \exp((L\cap P_i)\wedge P_i)$$
$$\times \exp(\frac{L\cap P_i}{[L\cap P_i,P_i]})\ \exp(\gamma^*_{c+1}(S_i))\,.$$

The bound of Theorem B(i) now follows from Propositions 9 and 11. (We take $N = G = S_i$ in Proposition 11.)

To prove Theorem B(ii) it suffices to note that the condition $[[P^{p^{i-1}},P],P] \subset P^{p^i}$ is equivalent to saying that the pair $(P^{p^{i-1}}/P^{p^i},\ P/P^{p^i})$ has nilpotency class at most 2.

6. Proof of Theorem C

Consider the dihedral group $D_n = \langle a,b \mid a^2 = b^n = (ab)\rangle$ with $n = 2^r m$ where $m \geq 1$ is odd. The smallest term of the lower central series of D_n is $L = \gamma_r(D_n) \cong C_m$. Proposition 7(ii) implies that the relative multiplier $\mathcal{M}(L,G)$ is trivial. Proposition 6 therefore implies an isomorphism $\bigotimes^{c+1}(L,G) \cong L\otimes G$ for all $c \geq 1$. So Proposition 10 yields a commutative diagram

$$\begin{tikzcd}
 & \mathcal{M}(L,D_n)=1 \arrow[r] \arrow[d, tail] & M^{(c)}(D_n) \arrow[r, "\cong"] \arrow[d, tail] & M^{(c)}(D_{2^r}) \arrow[r] \arrow[d, tail] & L/\gamma_{c+1}(L,D_n)=1 \\
 & L\otimes D_n \arrow[r] \arrow[d, two heads] & \gamma^*_{c+1}(D_n) \arrow[r] \arrow[d, two heads] & \gamma^*_{c+1}(D_{2^r}) \arrow[r] \arrow[d, two heads] & 1 \\
1 \arrow[r] & C_m \arrow[r] & \gamma_{c+1}D_n \arrow[r] & \gamma_{c+1}D_{2^r} \arrow[r] & 1
\end{tikzcd}$$

in which the rows are exact and the columns are short exact. From this, and the isomorphism $\gamma_{c+1}D_n \cong C_m \times \gamma_{c+1}D_{2^r}$, we derive the isomorphism

$$\gamma^*_{c+1}(D_n) \cong C_m \times \gamma^*_{c+1}(D_{2^r})\,.$$

A description of $\gamma^*_{c+1}(D_{2^r})$ is given in [16]. Alternatively, the description can be re-obtained as follows. The central extensions $C_2 \hookrightarrow D_{2^r} \twoheadrightarrow D_{2^{r-1}}$ and repeated applications of Proposition 15(i) imply that $|\gamma^*_{c+1}(D_{2^r})| \geq |\gamma^*_{c+1}(C_2 \times C_2)| \times 2^{r-1}$. These central extensions together with repeated applications of Proposition 15(iii) imply that $\gamma^*_{c+1}(D_{2^r})$ has the same number of generators as $\gamma^*_{c+1}(C_2 \times C_2)$, namely one generator for each basic commutator on two generators a, b. Theorem B implies that $\exp(\gamma^*_{c+1}(D_{2^r})) \leq 2^r$. So, to obtain the isomorphism $\gamma^*_{c+1}(D_{2^r}) \cong C_{2^r} \oplus (C_2)^{\chi_{c+1}(2)-1}$ it suffices to verify that at least all but one of the generators are of order 2, and that the generators a, b act as stated in the theorem. (Note that the invariant $\gamma^*_{c+1}(G)$ is abelian precisely when $\gamma_{c+1}(G)$ acts trivially on it.) This verification, which we leave to the reader, yields the desired description of $\gamma^*_{c+1}(D_{2^r})$ and completes the proof of part (i) of the theorem.

To prove part (ii) we note that the quaternion group Q_n is not of the form $H/Z(H)$ for any group H with centre $Z(H)$ [3]. Proposition 12(ii)(iii) thus implies that the cth term $Z^*_c(Q_n)$ of the upper epicentral series of Q_n contains the centre $Z(Q_n) \cong C_2$. Since $Q_n/Z(Q_n) \cong D_n$, the isomorphism $\gamma^*_{c+1}(Q_n) \cong \gamma^*_{c+1}(D_n)$ follows from Proposition 12(iv).

To prove part (iii) we note that an extraspecial group $E(p,k)$, $k \geq 2$, is not of the form $H/Z(H)$ for any group H with centre $Z(H)$ [3]. Arguing as in the previous paragraph, we see that $\gamma^*_{c+1}(E(p,k)) \cong \gamma^*_{c+1}(E(p,k)/Z(E(p,k))) \cong \gamma^*_{c+1}(C_p \times C_p)$. Proposition 13 completes the proof of part (iii). This argument also holds for $E(p,1)^-$, $p \geq 3$.

To obtain our partial description of $\gamma^*_{c+1}(E(p,1)^+)$, $p \geq 3$, we first remark that $Z^*_1(G)$ is trivial for the group $G = E(p,1)^+$ [3]. Letting $Z = Z(G)$ denote the centre of this group, Proposition 10 and Proposition 1(v) yield an exact sequence $\bigotimes^{c+1}(Z,G) = Z \otimes G^{ab} \otimes \cdots \otimes G^{ab} \to \gamma^*_{c+1}(G) \xrightarrow{\phi} \gamma^*_{c+1}(C_p \times C_p) \to 1$. The group $\bigotimes^{c+1}(Z,G)$ is elementary abelian of rank 2^c, and Proposition 12(iv) implies that ϕ has non-trivial kernel. Theorem B implies that $\exp(\gamma^*_{c+1}(G)) = p$. The commutator subgroup $[G,G] = Z$ acts trivially on $\gamma^*_{c+1}(G)$, and so $\gamma^*_{c+1}(G)$ is abelian. Hence $\gamma^*_{c+1}(G)$ is elementary abelian of rank at most $\chi_{c+1}(2)+2^c$, and at least $\chi_{c+1}(2)+1$.

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